Local Analysis of Linear Networks
by Aggregation of Characteristic Lines

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Abstract
Model-based systems such as GDE require local reasoning methods in order to perform dependency recording during model analysis. For linear networks, this poses a well known problem concerning the completeness of the derived predictions: naive component-oriented methods will often derive weaker predictions than actually possible. This is due to the cyclic computational dependencies (algebraic loops) that usually occur in the simultaneous equations describing the network’s steady-state behavior. These cycles tend to halt constraint propagation. We present an algorithm to derive additional constraints from a given resistive network based on series-parallel-star (SPS) aggregation. The resulting constraint net contains no cyclic dependencies. Hence, local propagation of value restrictions suffices to derive values for voltage drops and currents. The presented approach works for arbitrary linear networks with one source. A straightforward generalization to networks with multiple sources and non-linear network elements is described. SPS analysis has been implemented and applied to qualitative network analysis for model-aided decision tree development for the diagnosis of car electrical circuits. It may be applied as well to model-based diagnosis, failure mode and effects analysis (FMEA) or related application fields.

Introduction
Since the advent of model-based diagnosis in the early eighties considerable progress has been achieved in the logical foundation of diagnosis and - building on the seminal work of de Kleer and Williams [6] - in the development of sound and complete algorithms that realize the model-based paradigm. Although central for any model-based technique, surprisingly little effort has been spent on the development of complete methods for automated model analysis. As de Kleer and Williams remark in [6]: "Yet a logically complete inference strategy does not guarantee that the set of predictions will also be complete. For example in an analog domain, producing exact predictions often involves solving systems of higher order non-linear differential equations. As this type of equation is not generally solvable with known techniques, completeness in the predictor is currently beyond our reach. In practice propagation of constraints (without symbolic algebra) provides a good compromise between completeness and computational expense." Here, de Kleer and Williams referred to the automated analysis of electrical circuits that contain non-linear components like diodes or transistors. However, today, even the automated and complete analysis of simple resistive networks, i.e. of networks consisting of ideal sources and resistors, with ATMS-compatible i.e. with local methods, is currently not state of the art, cf. e.g. [7], [2], [12], who all present efficient but incomplete solutions to the problem.

On the other hand, resistive networks are a useful tool for modeling a wide range of technical devices. Due to the analogy between the physical laws relating voltage, pressure, force, and torque on one hand, and current, volume flow, velocity, and angular speed on the other (cf. [5]), such networks can be used to model general flow- and effort phenomena that are present in virtually all electrical, hydraulic, mechanical, and thermodynamic devices. Examples of components whose steady-state behavior can be modeled using linear or non-linear network elements are: electrical wires, bulbs, coils, switches, diodes, batteries, hydraulic pipes, oil filters, valves, pumps, mechanical rotating shafts, clutches, and motors.

This paper develops an efficient and local method for the automated analysis of linear networks. The paper is organized as follows: In the next Section we recall the notion of SPS trees as introduced in [9] and list some applications. Then we present the generalisation of SPS analysis to general linear elements given by characteristic lines. This generalisation allows the local analysis of linear networks with multiple sources and - hence - of linear dynamical systems. Further, for series-parallel reducible networks, non-linear elements can be handled as well.

SPS-Analysis of Resistive Networks
In this paper, we assume that each single component of the device under consideration is modeled by a resistive network (often a single resistor). The network modeling the entire device is then generated by connecting these component models according to the structural description of the device. Further, we assume that the component
models translate all relevant assumptions about faulty or operational states of the device components into assignments for the resistance and source parameters of the network. That is, for analyzing a network, we assume that all or some of the resistances and the voltage (or current) of the ideal voltage (or current) source are known (exogeneous) and that we are looking for the resulting currents and voltages. We first focus on linear networks with one source and treat the more general case later.

**Series-Parallel Trees**

Figure 1 shows a resistive network with three resistors \( R_1, R_2, R_3 \) and an ideal voltage source \( U_0 \) with its corresponding series-parallel tree (SP tree), i.e. the tree resulting from series-parallel reduction of the network. The arrows in the network graph define positive direction of current. In the tree, \( R_2 \) and \( R_3 \) are replaced by an equivalent resistor \( PA_2 \). \( PA_1 \) is in series with \( R_1 \), hence, both are replaced by an equivalent resistor \( SE_2 \) representing the total resistance of the network with respect to the given voltage source \( U_0 \).

In this paper, \( r_j \), \( u_k \), and \( i_k \) always denote resistance, voltage drop, and current of the resistor with index \( k \). For instance, in Figure 1, \( r_i \) is the resistance of resistor \( SE_i \) and \( u_i \) is the voltage drop over resistor \( R_i \).

**Figure 1: Resistive network and its SP tree**

Knowing the SP tree of a network is quite useful. It enables the straightforward calculation of numerical, qualitative, or algebraic expressions for all unknown quantities of the network. Each node in the SP tree is associated with certain constraints as shown in the table in Figure 7. Hence, a SP tree can be interpreted as a constraint net. Figure 2 shows the constraint net corresponding to the SP tree given in Figure 1. Beside the common operators +, ×, /, we use the parallel operator \( r_j || r_k \) = \( r_j \times r_k (r_j + r_k) \). The arrows in Figure 2 indicate the causal ordering (cf. [3]) of the variables when \( r_j, r_k, r_m, \) and \( u_n \) are exogeneous, i.e. given.

As one can see, the causal ordering first aggregates the given resistances until the total resistance \( r_n \) of the circuit is known. Then, the known voltage \( u_n \) of the source is used to compute first the total current \( i_n \), and from this, all remaining currents and voltage drops in the network. Note that there are no cyclic dependencies in the constraint net. Hence, local propagation of value restrictions will suffice to derive values for all the unknowns. In general, the SP tree of a circuit yields a cycle-free decomposition of the network’s constraint structure: the SP tree “breaks” all the cyclic dependencies introduced by the simultaneous equations by introducing new variables associated with the aggregate resistors.

**Figure 2: Constraints, arrows indicate causal ordering**

**Series-Parallel-Star Trees**

Unfortunately, not every network can be reduced using series and parallel reductions only. Figure 3 gives a minimal example for such an irreducible network.

**Figure 3: A bridge circuit and its SPS tree**

In this context, a key observation is: If in a resistive network neither series nor parallel reductions are possible, then all vertices of the network graph, except possibly the start and end of the source, have an edge degree > 2, i.e. every vertex of the network constitutes the center of a star with 3 or more branching resistors. Hence, if there would be a reduction rule for stars with 3 or more branches, we could develop a reduction procedure for arbitrary networks. Such a star reduction rule exists in fact and is known to electrical engineers as star-mesh conversion, cf. [11].

**Figure 4: Examples of the star reduction rule**

Figure 4 shows some instances of the star reduction rule. The rule states that a star of \( n \) resistors, \( n > 2 \), branching from a central vertex \( v_0 \) can be replaced by \( n (n-1) / 2 \) equivalent resistors. The replacing resistors can always be chosen such that the currents entering and leaving the star equal the currents entering and leaving the mesh, i.e. seen “from outside”, the star and the replacing mesh are electrically equivalent. At first glance, this replacement...
does not look like a simplification of the network, because for \( n > 3 \) the number of resistors increases. On the other hand, the application of the star reduction rule removes the central vertex \( v_0 \) without introducing new vertices, so, measured in the number of vertices, the network graph is simplified. Vertex removal corresponds to variable elimination in the Gauss algorithm for solving a system of linear equations.

![Figure 5: SPS Reduction of a bridge circuit](image)

Figure 5 shows the reduction of a bridge circuit using series, parallel, and star reduction resulting in the series-parallel-star tree (SPS tree) shown in Figure 3. First, the star \( R_j, R_k, R_m \) around vertex \( v_0 \) is replaced by the mesh \( R_{jk}, R_{km}, R_{jm} \). The resulting network is then further reduced using series and parallel reductions only.

A close look at the relations between the resistors of a star and the replacing resistors of the equivalent mesh shows that we can still express all currents and voltage drops in the star in terms of currents and voltage drops in the replacing mesh, and that the resistances of the \( n(n-1)/2 \) resistors in the mesh can be expressed in terms of the \( n \) star resistances. In other words: The local calculation schema sketched for SP trees in the previous Section works as well for SPS trees.

![Figure 6: Constraints for the current through the bridge resistor \( R_2 \)](image)

For example, the SPS tree from Figure 3 yields constraints for the current \( i_2 \) through the bridge resistor \( R_2 \) as shown in Figure 6. Again, the directions of the arcs show the causal ordering if the resistances and the voltage of the source are regarded as exogeneous variables. In Figure 6, beside the operators \(+, -, \times, \div, \| \), we use the star-conversion operator

\[
f_\ast(r_j, r_k, r_m) = r_j r_k (\frac{1}{r_j} + \frac{1}{r_k} + \frac{1}{r_m}).
\]

Note again that there are no cyclic dependencies among the variables, i.e. local propagation will in fact derive a prediction for the current \( i_2 \).

<table>
<thead>
<tr>
<th>END-BRANCH</th>
<th>SELF-LOOP</th>
<th>SOURCE</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e_1 )</td>
<td>( e_1 )</td>
<td>( -\frac{e_0}{e_1} )</td>
</tr>
<tr>
<td>( EB )</td>
<td>( SL )</td>
<td>( S )</td>
</tr>
<tr>
<td>( i_1 = 0 )</td>
<td>( u_1 = 0 )</td>
<td>voltage source: ( u_i = u_0 ) current source: ( i_i = i_0 )</td>
</tr>
</tbody>
</table>

![Figure 7: Constraints holding in SPS trees](image)

The table in Figure 7 summarizes the algebraic relations needed to interpret a SPS tree as a constraint network. The table contains two more reduction types, namely self-loop \( SL \) and end-branch \( EB \). A self-loop can be regarded as a zero-voltage source, and an end-branch is a zero-current source. \( SL, EB, \) and ideal source \( S \) are the three possible roots of a SPS tree.

![Figure 8: Short circuit (dotted line) that causes a bridge](image)

At first glance, star reduction may seem to be merely of theoretical interest, since bridge circuits involving stars do rarely occur in most applications (at least in the car electrical circuits of our application). This may be quite different, if structural errors such as short circuits or leakages are considered. As a motivation, Figure 8 gives a minimal example illustrating the probably surprising fact that a simple series-parallel reducible network can become SP irreducible in the presence of a short circuit to ground.
Sign Conventions in SPS trees

Inspired by bondgraphs (cf. [5]), we use the following sign conventions to define positive current direction in SPS trees:

- A down-arrow from a SE or PA resistor $R$ down to a replaced resistor $R_i$ means: $R$ and $R_i$ have the same sign convention. Correspondingly, an up-arrow encodes complementary sign conventions for $R$ and $R_i$.
- For the $n$ resistors of a star around vertex $v_n$ the following convention holds: Down-arrow to a star resistor $R$ in the SPS tree means: current through $R$ to $v_n$ counts positive, and up-arrow means: current through $R$ away from $v_n$ counts positive.
- The $n(n-1)/2$ mesh resistors replacing a $n$-star are by convention directed as follows: Let for $1 \leq j < k \leq n$ $R_{jk}$ be the resistor connecting vertex $v_j$ with vertex $v_k$. Then $R_{jk}$ is by convention directed from $v_j$ to $v_k$, i.e. current in this direction counts positive.

End-branch $EB$ and self-loop $SL$ do not require sign conventions, because the current through a resistor marked $SL$ or $EB$ is always zero.

Derivation of the star-removal rule

In this Section, we derive the star removal rule, mainly in order to provide a basis for the extension of this rule to general linear elements presented later. The following derivation is based on [11].

For $1 \leq k \leq n$ and $n > 2$, let $r_i$ be the resistance of the star resistor $R_{ij}$, $i_j$ the current from $v_j$ through $R_i$ to the star center $v_n$, and $u_k = r_k i_k$ the corresponding voltage drop. For the voltage drop $u_{jk}$ between to vertices $v_j$ and $v_k$, we get

$$1 \leq j, k \leq n : \quad u_{jk} = u_j - u_k = r_j i_j - r_k i_k$$

Solving for $i_j$ yields

$$1 \leq j, k \leq n : \quad i_j = \frac{r_j i_j - u_{jk}}{r_k}$$

Because of Kirchhoff’s current law we have

$$1 \leq j \leq n : \quad \sum_{k=1}^{n} i_k = \sum_{k=1}^{n} \frac{r_j i_j - u_{jk}}{r_k} = 0$$

Solving this equation for $i_j$ gives us

$$1 \leq j \leq n : \quad i_j = \sum_{k=1}^{n} \frac{r_0}{r_j r_k} u_{jk}$$

with the so called star resistance $r_0$ defined as

$$1 = \sum_{k=1}^{n} \frac{1}{r_k}$$

Alternatively, the current $i_j$ entering vertex $v_j$ in direction to $v_n$ can be expressed using the unknown mesh resistances $r_{jk}$

$$1 \leq j \leq n : \quad i_j = \sum_{k=1}^{n} i_{jk} = \sum_{k=1}^{n} \frac{1}{r_{jk}} u_{jk}$$

Because star and mesh have to be electrically equivalent, both expressions for $i_j$ should be equal. Hence, comparison of these two expressions gives us

$$1 \leq j < k \leq n : \quad r_{jk} = \frac{r_j r_k}{r_0}$$

which is the only solution.

Qualitative interpretation of SPS trees

Often, we are interested only in a qualitative description of a network’s behavior. For this purpose, we define three sets of qualitative values, ordered with respect to $<$ as indicated below and representing qualitative resistance and signed qualitative current and voltage drop.

$$R = ( \text{ZERO}, \text{POS}, \text{INF} )$$

$$I = ( \text{MINF}, \text{NEG}, \text{ZERO}, \text{POS}, \text{INF} )$$

$$U = ( \text{MMAX}, \text{NEG}, \text{ZERO}, \text{POS}, \text{MAX} )$$

Voltage drops occurring in a resistive network are bounded by a finite value MAX or minus MAX = MMAX, as long as we are assuming an ideal voltage source. As a consequence, the current through a ZERO resistor might be infinite, i.e. INF or MINF. It is straightforward to derive qualitative abstractions of the relations given in Figure 7 using these qualitative value sets. For example, series and parallel aggregation abstracts to

$$[r_i + r_j] = \max([r_i],[r_j]) \quad [r_i \cdot r_j] = \min([r_i],[r_j])$$

where $[ ]$ is the abstraction operator and $\max$ (min) yields the maximum (minimum) of its qualitative arguments wrt. the qualitative $<$ relation defined above.

In star-mesh conversion, the algebraic relation between $r_j$, $r_i$ in the star and $r_{jk}$ in the mesh as derived above

$$r_{jk} = r_j r_k \sum_{m=1}^{n} \frac{1}{r_m}$$

abstracts in its qualitative version to

$$[r_{jk}] = \begin{cases} \text{INF}, & \text{if } \exists m: j \neq m \neq k \land [r_m] = \text{ZERO} \\ \max([r_i],[r_j]), & \text{else} \end{cases}$$

Hence, by replacing the algebraic relations in Figure 7 by their qualitative abstractions, we can turn SPS trees into a tool for qualitative network analysis. For SP reducible networks, the resulting qualitative analysis method is complete in the sense that - for given qualitative assumptions about resistances and the source - local propagation through the SP tree derives strongest predictions w.r.t. to the qualitative domains of the variables. However, for networks whose SPS tree involves at least one star-mesh conversion, the qualitative analysis is not complete. This is due to the formula for the current $i_j$ through the star resistor $R_s$, which is computed as difference of two sums that refer in part to the same variables, cf. Figure 7. Hence, additive cancelations may occur. Therefor, the qualitative abstraction of this formula can cause weaker predictions than actually possible, since additive cancellations cannot be handeled properly, due to
the missing inverse for qualitative addition ("selection problem").

**A Procedure for SPS-Reduction**

In this section we will present a procedure that derives a SPS tree for an arbitrary given resistive network.

The network graph under consideration is given here as a directed graph \( G = (E, V) \) where \( E \) is a finite set of directed edges and \( V \) is a set of vertices. There is exactly one edge \( s \in E \) representing the source. All edges \( E \setminus \{s\} \) represent resistors. The direction of an edge defines the direction in which current counts positive. \( E \) may contain self-loops (edge with same start and end vertex) and parallel edges. \( G \) is not necessarily connected, \( \text{degree}(v) \) is the number of different in- and outgoing edges of vertex \( v \) and \( \text{vertices}(e) \) denotes the set of start and end vertices of edge \( e \).

\[
\begin{align*}
\text{sps-reduce}(E, V, s) \\
(1) & \quad E' = E \setminus \{s\} \\
(2) & \quad V' = V \setminus \text{vertices}(s) \\
(3) & \quad \text{edge-reduce}(E') \\
(4) & \quad \text{while } V' \text{ not empty} \\
(5) & \quad v = \text{vertex with minimal degree in } V' \\
(6) & \quad \text{case } \text{degree}(v) = \\
(7) & \quad 0 : \text{isolated-vertex. do nothing} \\
(8) & \quad 1 : \text{end-branch. replace the edge by } EB \\
(9) & \quad 2 : \text{series. replace both edges by } SE \\
(10) & \quad \geq 3 : \text{star. replace } n \text{ star edges} \\
& \quad \text{by } n (n-1)/2 \text{ mesh edges} \\
(11) & \quad \text{end case} \\
(12) & \quad \text{remove } v \text{ from } V' \\
(13) & \quad \text{edge-reduce}(E') \\
(14) & \quad \text{end while} \\
\text{edge-reduce}(E) \\
(1) & \quad \text{for each edge } e_1 \in E \\
(2) & \quad \text{case } e_1 \\
(3) & \quad \text{is a self-loop: replace } e_1 \text{ by } SL \\
(4) & \quad \text{is parallel to } e_2 \in E: \text{ replace } e_1, e_2 \text{ by } PA \\
(5) & \quad \text{end case} \\
(6) & \quad \text{end for}
\end{align*}
\]

**Figure 9: The SPS reduction procedure**

Figure 9 presents the algorithm for building a SPS tree. The procedure terminates after \(|V'| \) iterations through the while loop, because every iteration removes one vertex at (12) and none of the replacement operations within the loop introduces a new vertex. After termination of \(\text{sps-reduce} \), we have \( E' = \{e \} \) and \( e \) represents the total resistance of the network w.r.t. source \( s \) or \( E'' = \{ \} \) and the network is an open circuit w.r.t. source \( s \).

If a network can be analyzed using series and parallel aggregations only, i.e. without star conversion, then \(\text{sps-reduce} \) will discover such an aggregation sequence. This is due to the ordering of the vertices by degree at line (5).

A SPS tree, once derived, enables network analysis using local propagation only. But even the procedure \(\text{sps-reduce} \), i.e. the derivation of SPS trees, can be implemented using local information only, as e.g. the local degree of a vertex. In an ATMS-based system, a justification for a derived equivalent resistor will then refer to the topology of the network. Actually, we not only get justified values, but also topological justified constraints. During diagnosis, a conflict involving such a topological justification will then contain information about possible structural faults, e.g. short circuits. For example, a series resistor \( SE \) replacing two resistors \( R_{s-v} - R_v \) is valid, as long as \( R_1 \) and \( R_2 \) are valid resistors and vertex \( v \) is not connected to any other vertex (by e.g. an unknown structural fault). This issue is subject of further research.

The idea to use series-parallel reductions is not new. [13] use equivalent resistors called "slices" to generate additional constraints for local network analysis. [7] and [2] use series-parallel reduction in a similar manner. The specific contribution of this paper rests on the following insights:

- series-parallel reduction plus star-mesh conversion constitutes an analysis method for arbitrary networks,
- the resulting SPS trees yield cycle-free computation sequences for all quantities in the network,
- hence, local propagation through the SPS tree suffices to derive values for all these quantities.

Thus, for the special case of resistive networks, SPS trees solve the problem that simultaneous equations usually cause (see [4]) when processed by local propagation of value restrictions.

**Applications**

We have implemented SPS analysis as described so far and have applied this to the model-aided derivation of decision trees for diagnosis of car electrical circuits. We used qualitative abstractions of the SPS relations as given in Figure 7, mainly because in our scenario, exact quantitative information is not available. The application scenario is described in [8] and [10].

If we replace the real-valued relations in Figure 7 by their complex-valued extensions, (i.e. real-valued resistances turn into complex-valued, frequency-dependent impedances) we can use SPS trees to perform steady-state analysis of RLC-networks, i.e. of networks containing an AC-source, resistors, coils, and capacitors. [2] use SP trees in that manner. However, since they do not use star conversion, their tool is restricted to SP-reducible RLC-networks.

Building on [9], [1] use SPS trees as a tool for diagnosing a heavy fuel oil transfer system of a container ship based on qualitative deviational models. In this model-based diagnosis system, SPS trees replaced the previously used bondgraph [5] models. Their work exploits another interesting feature of SPS trees: A SPS tree can be interpreted as a hierarchy of stepwise simpler, but behaviorally equivalent networks. This can be used by problem solvers to choose the most simple network.
sufficient for solving the given problem in order to reduce problem complexity.

Aggregating characteristic lines

SPS analysis as presented and implemented so far is restricted to linear networks with at most one source. In a more general approach, we will now unify resistor and source by representing both elements by their characteristic lines in the voltage-current diagram.

A linear characteristic line is a tuple \( h = (r, U, I) \) with \( \text{dom}(h) = H \subseteq \mathbb{R}_+^* \) and

\[
H = \{ (r, U, I) \mid U = - r I \}
\]

where \( U \) and \( I \) represent the intersection points of the characteristic line with the \( u \)- and \( i \)-axis as illustrated in Figure 10 and \( \text{IR} = IR \cup \{ \pm \infty \} \). The representation as a triplet is redundant - a pair of variables would suffice - but more convenient for further analysis.

**Figure 10:** A linear characteristic line \( h = (r, U, I) \)

Instead of aggregating resistances in a SPS tree, we now aggregate characteristic lines. For series and parallel aggregation, this is easily done by simply adding the lines as illustrated in Figure 11.

**Figure 11:** Series aggregation of two characteristic lines

The table in Figure 12 gives the constraints \( SE(h_x, h_y, h_z) \) and \( PA(h_x, h_y, h_z) \) defining the relation between two linear elements \( h_x \) and \( h_y \) connected in series and parallel, resp., and the equivalent element \( h_x \). For example, if an element \( h = (r, \neq \infty, U, I) \) is in series with an element \( h = (r, \neq \infty, U, I) \), both elements can be replaced by an equivalent element \( h = (r + r, U + U, I) \) where \( I \) is uniquely defined by \( r \). The relation \( H \). Aggregation of characteristic lines can uncover an inconsistency in the network graph, denoted by \( \perp \). For example, two ideal current sources \( I \neq I \) connected in series represent an unsatisfiable situation, as well as to ideal voltage sources \( U \neq U \) connected in parallel.

For star-mesh conversion, the aggregation rule is not that straightforward, but exists too. We assume that the star contains \( m \) elements, \( m > 2 \). For \( 1 \leq k \leq n \), the elements are represented as a series of a linear resistor \( r_k \) and an ideal voltage source \( U_k \). For \( n < k \leq m \) the star elements are ideal current sources, i.e. \( n = m \). We need a rule to replace every pair of linear star elements \( h \) and \( h \) by an equivalent mesh element \( h \). To derive formulas for the \( h \), we have two consider two cases.

1) There is at least one element in the star not being an ideal current source, i.e. \( n > 0 \). By generalizing the derivation given above for this case, we get for the voltage drop \( u \) between two vertices \( v \) and \( v \):

\[
1 \leq j, k \leq n: \quad u = r_i - r_i + U - U
\]

Solving this equation for \( i \) yields

\[
1 \leq j, k \leq n: \quad i = \frac{r_i - u + U - U}{r_k}
\]

Because of Kirchhoff’s current law we have

\[
\sum_{k=1}^{n} I_k + \sum_{k=1}^{n} i_k = 0
\]

Substituting \( i \) gives us

\[
1 \leq j \leq n: \quad \sum_{k=1}^{n} I_k + \sum_{k=1}^{n} r_j i_j - \sum_{k=1}^{n} u_k U - U = 0
\]

Dividing this equation by \( r \) and solving for \( i \) gives us

\[
1 \leq j \leq n: \quad i = \frac{r_j r_k}{r} (u_k - U + U)
\]

with the so called star resistance \( r \) as defined above. Alternatively, the current \( i \) entering vertex \( v \) in direction to \( v \) can be expressed using the unknown mesh elements \( h \):

\[
1 \leq j \leq n: \quad i = \frac{r_j r_k}{r} (u_k - U - U)
\]

Because star and mesh have to be electrically equivalent, both expressions for \( i \) should be equal. Comparison of the two expressions gives us

\[
1 \leq j < k \leq n: \quad r_j = r_k = \frac{r_j}{r_k}
\]

\[
1 \leq j < k < m: \quad r_j = r_k = \infty
\]

This defines all the mesh elements \( h \), except those with \( n < j < k \leq m \). To find out how these missing elements contribute to the currents in the mesh, we look at the current contributed by the mesh elements defined so far. For \( k > n \), we consider the current \( i \) entering the star at vertex \( v \). We can conclude that the missing elements contribute no current at all, i.e.

\[
n < j < k \leq m: \quad r_j = r_k = \infty, \quad I_j = 0
\]
2.) Let us now consider the second case that all the star elements are ideal current sources, i.e. \( n = 0 \). There are two subcases.

2.1) If the sum of all the \( m \) currents is different from 0, due to Kirchhoff's current law, the network is inconsistent, denoted by \( \perp \).

2.2) Else, we can choose an arbitrary \( q \) with \( 1 \leq q \leq m \), remove element \( I_t \) from the graph and identify nodes \( v_t \) and \( v_j \). This actually replaces the star of \( m \) ideal current sources by an equivalent star of \( m - 1 \) ideal current sources. In contrast to the cases considered before, in case 2.2, the mesh is not uniquely defined by the given star.

<table>
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<tr>
<th>( SE )</th>
<th>( r_1 \neq \pm \infty )</th>
<th>( r_1 = \pm \infty )</th>
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<td>( \perp ) if ( I_1 \neq I_2 ) else ( I_0 = I_1 = I_2 )</td>
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<td></td>
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<td>( r_0 = \pm \infty )</td>
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<table>
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<td>( r_2 = 0 )</td>
<td>( U_0 = U_2 )</td>
<td>( \perp ) if ( U_1 \neq U_2 ) else ( U_0 = U_1 = U_2 )</td>
</tr>
<tr>
<td></td>
<td>( r_0 = 0 )</td>
<td>( r_0 = \pm \infty )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( STAR )</th>
<th>( r_j \neq \pm \infty )</th>
<th>( r_j = \pm \infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r_k \neq \pm \infty )</td>
<td>( U_{jk} = U_j - U_k )</td>
<td>( I_{jk} = \frac{I_j}{r_k} )</td>
</tr>
<tr>
<td></td>
<td>( r_j = \pm \infty )</td>
<td>( r_k = \pm \infty )</td>
</tr>
</tbody>
</table>
| \( r_k = \pm \infty \) | \( r_j = \pm \infty \) | \( \perp \) if \( \Sigma I_k \neq 0 \)
| | else choose \( q \) | \( I_{jk} = \frac{I_j}{r_j} \) if \( k = q \)
| | | \( -I_j \) if \( j = q \)
| | | 0 else |

**Figure 12: Aggregation of linear characteristic lines**

To summarize, the table in Figure 12 gives the constraint \( STAR \) that - together with \( H \) - defines the relation between a star of linear elements and the behavioral equivalent mesh.

**Application to linear dynamical systems**

With the extension to general linear elements, networks with arbitrary numbers of voltage and current sources can be analyzed in a local way. As an application, we can now analyze states of linear dynamical systems (containing energy-storing elements), because, within a time slice, each energy-storing element can be represented by an ideal source element.

**Figure 13: RC circuit and its SP tree**

Consider as an example the RC circuit shown in Figure 13. At time point \( t \), the capacitor \( C \) with known capacity \( C \) and known load \( q(t) \) behaves like an ideal voltage source with voltage \( U = q(t) \). Using aggregation of the characteristic lines, SPS analysis will first build the SP tree shown in Figure 13. At the root of that tree - an end branch \( EB \) - the current is known to be zero. This information propagates backwards through the SP tree determining thereby the operating points of all the characteristic lines. Hence, finally the current \( i_e(t) \) into the capacitor is known, which in turn yields the capacitor’s load at the time step \( t + 1 \) as \( q_e(t + 1) = q_e(t) + i_e(t) \Delta t \).

**SP-Analysis of non-linear systems**

Another application of the generalisation to characteristic lines is the treatment of piecewise-linear or non-linear elements. A network element might be given by a non-linear characteristic line, to represent e.g. a diode, a hydraulic check valve, or a motor whose speed depends on the torque in a non-linear way. We can aggregate such non-linear lines as well. This works as before for series and parallel aggregation by simply adding the lines.

Star-mesh conversion of non-linear elements is an open issue. In the following, we sketch the conversion for a star of piecewise-linear elements. Assume, all the \( n \) star elements are represented by piecewise-linear characteristic lines \( h_i \), each line composed of \( m \) linear sections \( (h_i_1, \ldots, h_i_m) \), each section \( h_i_q \) defined on a certain voltage interval \( u_{iq} \). Since \( n - 1 \) currents in the star can be chosen independently, the star is always in one of \( m^n \) operational states, i.e. there are \( m^n \) possible \( n \)-tuples of linear sections containing the \( n \) operation points of the \( n \) star elements. For each such tuple of linear sections, we can compute an equivalent mesh as shown above. For each mesh element \( h_j \), this gives us \( m^n \) linear sections. Each section is defined on the voltage interval \( u_{ij} - u_{iq} \), where \( u_j \) and \( u_q \) are the definition intervals of the linear sections \( h_{ij} \) and \( h_{iq} \).
Conclusion

We presented a method for local analysis of arbitrary resistive networks with one source. Such networks can be used for modeling a wide range of electrical, hydraulic, pneumatic, mechanical, and thermodynamic devices. The analysis proceeds in two stages: First, the network is reduced to a single equivalent resistance w.r.t a given source by replacing series, parallel and $n$-stars, $n \geq 3$, by equivalent resistors. The resulting so-called SPS tree represents the network as a hierarchy of stepwise simpler equivalent networks. This hierarchical representation might be of great use for the treatment of very large networks if combined with hierarchical multi-level problem solvers. The derivation of a SPS tree was not formulated as a local analysis method here, although this is possible. Instead, a global graph-analysis procedure $sps$-$reduce$ was given for this task. In a second step, the SPS tree is interpreted as a constraint network that enables the prediction of all quantities in the network by local propagation of value restrictions through the SPS tree.

The presented analysis technique has been implemented and applied to qualitative electrical network analysis for model-aided decision tree development. It may be applied to model-based diagnosis, failure mode and effects analysis (FMEA) or related application fields as well. We generalized SPS-analysis to network elements described by linear characteristic lines in the voltage current diagram and showed how this can be used to analyze linear dynamical systems in a local way. The proposed extension works even for non-linear elements, at least for networks that are series-parallel reducible.

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References


